On the existence of balanced solutions in optimal economic growth and investment problems

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T growth models
T investment

Research memorandum

TILBURG INSTITUTE OF ECONOMICS
DEPARTMENT OF ECONOMETRICS

On the existence of balanced solutions in optimal economic growth and investment problems.

Joop J.M. Evers

1. One-sector model of capital accumulation: an example.

We consider an economy with two factors of production, capital and labor, that are combined to produce a single homogeneous output. The production is executed in a sequence of periods of equal duration. The input, consisting of capital and labor, received at the beginning of a period results in an output which is available at the end of that period.
At that moment the output of the preceeding period may be allocated, entirely or partly, to consumption or to investment in capital accumulation. Once invested the capital stock is not longer a good that is suitable for consumption.

Let $K(t)$ and $L(t)$ denote the stocks of capital and labor at the beginning of period t, then the amount of output available at the end of that period $Y(t)$ can be expressed by:

$$Y(t) = F[K(t), L(t)], \tag{1.1}$$

where $F[\cdot,\cdot]$ are the production functions which are:

(a) Homegeneous of degree one (in economic terms: constant returns to scale in capital and labor).

(b) Increasing in capital and labor.

(c) Concave.

Let $C(t)$ and $Z(t)$ denote the consumption and the investment allocated at the beginning of period t, then the fact that no more assumption and investment can be allocated than the quantity of output generated by production in the preceeding period, is expressed by the following inequality:

$$C(t) + Z(t) \leqq Y(t-1). \tag{1.2}$$

If capital is subject to evaporative decay such that $\mu K(t)$, with $\mu]0,1[$, is the remaining part of $K(t)$ at the end of period t, then the capital stock over next period is bounded by:

$$K(t+1) \leqq Z(t+1) + \mu K(t). \tag{1.3}$$

With respect to the amount of labor, we assume that population growth is independent of the economic variables, in such a manner that: $N(t+1) = \rho N(t)$, $N(t)$ being the population at the beginning of period t and ρ being a constant growth factor. So, starting with period $t = 0$, the population size may be expressed by:

$$N(t) = \rho^t N(0), \; t = 0,1,\ldots, \; \rho > 0. \tag{1.4}$$

Further, we assume that the number of able-bodied workers is a fixed fraction $\alpha \in [0,1]$ of the total population, so that the use of labor is bounded by:

$$L(t) \leqq \alpha N(t). \tag{1.5}$$

Putting (1.1) to (1.5) together, the following system of inequalities appears:

$$\left.\begin{array}{l} C(t+1)+Z(t+1)-F[K(t), L(t)] \leqq 0 \\ K(t+1)-Z(t+1)-\mu K(t) \leqq 0 \\ L(t+1) \leqq \rho^{t+1}\alpha N(0) \end{array}\right\} t = 0,1,\ldots \tag{1.6}$$

Defining for every period the quantities:

- aggregate capital per worker: $k(t) := K(t)/\{\rho^t \alpha N(0)\}$
- consumption per worker : $c(t) := C(t)/\{\rho^t \alpha N(0)\}$
- investment per worker : $z(t) := Z(t)/\{\rho^t \alpha N(0)\}$
- fraction productive workers : $l(t) := L(t)/\{\rho^t \alpha N(0)\}$,

and using the constant returns to scale property (viz. 1-a) of the production function, the restrictions (1.6) take the following form:

$$\left.\begin{array}{l} c(t+1)+z(t+1)-(\frac{1}{\rho})F[k(t),l(t)] \leqq 0 \\ k(t+1)-z(t+1)-(\frac{\mu}{\rho})k(t) \leqq 0 \\ l(t+1) \leqq 1 \end{array}\right\} t = 0,1,... \qquad (1.7)$$

Using for every period t the same concave objective function p[$\cdot$], a planning board is supposed to maximize the function:

$$\sum_{t=1}^{T} \pi^{t} p[c(t)], \qquad (1.8)$$

which implies the use of a planning horizon T and a time discount factor $\pi \in]0,1[$.
So, the problem consists of the finding of a path $\{(c(t), z(t), k(t), l(t))\}_{1}^{T}$ maximizing (1.8) subject to the inequalities (1.7), given the initial state $k(0) := \overset{0}{k}$, $l(0) := \overset{0}{l}$.
It may also be possible that at planning period T some terminal conditions are imposed. For instance: the terminal capital stock per worker k(T) is not less than some prescribed minimum $\underline{k}$:

$$k(T) \geqq \underline{k}. \qquad (1.9)$$

Assuming that this economy does no cease to exist, the fixation of the planning horizon T together with the implementation of terminal conditions introduces a certain arbitrariness. For that reason, it can be useful to operate with an infinite horizon, i.e. to look for a path $\{(c(t), z(t), k(t), l(t))\}_{1}^{\infty}$ that maximizing:

$$\sum_{t=1}^{\infty} \pi^{t} p[c(t)] := \lim_{T\to\infty} \sum_{t=1}^{T} \pi^{t} p[c(t)], \qquad (1.10)$$

subject to (1.7) with given initial state $k(0) = \overset{0}{k}$, $l(0) = \overset{0}{l}$.

2. General structure of optimal economic growth models.

More general growth and investment models with, for instance, several types of output, with many production factors, with multi-period-input-output production functions and a changing technology, and so on, can be characterized by a system of inequalities:

$$\left.\begin{array}{l} B[x(t+1);t+1] - A[x(t);t] + y(t+1) = f(t+1) \\ \\ x(t),\ y(t+1) \geqq 0 \end{array}\right\}\ t = 0,1,\ldots \qquad (2.1)$$

together with a sequence of objective functions:

$$\left\{ \sum_{t=1}^{T} \pi^t p[x(t);t] \right\}_{T=1}^{\infty}. \qquad (2.2)$$

The quantities of this system are specified as follows:

a) $\{x(t)\}_0^\infty$ a sequence of n-dimensional vectors, representing the "growth" path. The initial economic state vector $x(0)$ is supposed to be a fixed quantity.

b) $\{y(t)\}_1^\infty$ a sequence of m-dimensional <u>slack</u> vectors

c) $B[\cdot;t] : R_+^n \to R^m$, $t = 1,2,\ldots$, a sequence of <u>convex</u> functions

d) $A[\cdot;t] : R_+^n \to R^m$, $t = 0,1,\ldots$, a sequence of <u>concave</u> functions, such that, for every $x, y \in R_+^n$:

$$A[x+y;t] \geqq A[x;t],\ t = 0,1,\ldots \qquad (2.3)$$

e) $\{f(t)\}_1^\infty$ a <u>bounded</u> sequence of right-hand vectors

f) $p[\cdot;t] : R_+^n \to R^1$, $t = 1,2,\ldots$, a sequence of <u>concave</u> objective functions. The discount factor π is supposed to be a number in the open interval $]0,1[$.

Sequences $\{(x(t),y(t))\}_1^\infty \subset R_+^{n+m}$ which satisfy (2.1) for some $x(0) \in R_+^n$ will be called <u>feasible solutions</u>. Given the initial vector $x(0) := \overset{0}{x} \in R_+^n$, we shall call a feasible solution $\{(\hat{x}(t),\hat{y}(t))\}_1^\infty$ a <u>superior</u> solution with respect to $\overset{0}{x}$, if for the same initial vector <u>no</u> feasible solution $\{(\bar{x}(t),\bar{y}(t))\}_1^\infty$ exists, such that for some $\varepsilon > 0$ and some integer $S \geqq 1$:

$$\sum_{t=1}^{T} \pi^t p[\bar{x}(t);t] \geqq \varepsilon + \sum_{t=1}^{T} \pi^t p[\hat{x}(t);t], \quad T = S, S+1, \ldots \qquad (2.4)$$

Clearly, in this manner, the maximizing of an objective function (viz. §1) is replaced by a process of mutually comparization of feasible solutions with respect to a sequence of objective functions.

3. Duality relations.

As pointed out elsewhere (viz. ref. 7, §25) programming problem (2.1), (2.2), gives rise to the so called dual problem which can be formulated as seeking for a sequence of vectors $\{u(t)\}_1^\infty \subset R^m$ and a sequence of numbers $\{\mu(t)\}_1^\infty$, satisfying:

$$\left.\begin{array}{l} u(t)'B[z;t] - u(t+1)'A[z;t] + \mu(t) \geqq \pi^t p[z;t], \text{ for all } z \in R_+^n \\ u(t) \geqq 0, \ \mu(t) \geqq 0 \end{array}\right\} t=1,2,\ldots, \qquad (3.1)$$

and minimizing a "dual objective function"

$$u(1)'A[x(0);0] + \sum_{t=1}^{T} \{\mu(t) + f(t)'u(t)\}, \qquad (3.2)$$

or, in the infinite horizon case, minimizing:

$$u(1)'A[x(0);0] + \sum_{t=1}^{\infty} \{\mu(t) + f(t)'u(t)\}. \qquad (3.3)$$

Sequences $\{(\mu(t),u(t))\}_1^\infty$ which satisfy (3.1), will be called <u>feasible solutions</u> of dual problem. A sequence $\{(\hat{\mu}(t),\hat{u}(t))\}_1^\infty$ will be called a <u>dual superior solution</u> if he is feasible and if <u>no</u> dual feasible solution $\{(\bar{\mu}(t),\bar{u}(t))\}_1^\infty$ exists such that, for some $\varepsilon > 0$ and some integer $S \geqq 1$:

$$\bar{u}(1)'A[x(0);0] + \sum_{t=1}^{T} \bar{\mu}(t)+f(t)'\bar{u}(t) \leqq -\varepsilon + \hat{u}(1)'A[x(0);0] +$$

$$+ \sum_{t=1}^{T} \hat{\mu}(t)+f(t)'\hat{u}(t), \quad T = S, S+1,... \qquad (3.4)$$

Defining the sequence of functions $v[\cdot,\cdot,\cdot,\cdot;t] : R_{+}^{m+m+n+1} \rightarrow R^{1}$, $t = 1,2,...$ such that for all u, $w \in R_{+}^{m}$, $z \in R_{+}^{n}$, $\mu \in R_{+}^{1}$:

$$v[u,w,z,\mu;t] := u'B[z;t]-w'A[z;t]-\pi^{t}p[z;t]+\mu, \quad t = 1,2,..., \qquad (3.5)$$

straightforward calculations show that for any pair of feasible solutions $\{(x(t),y(t))\}_{1}^{\infty} \subset R_{+}^{n+m}$, $\{(\mu(t),u(t))\}_{1}^{\infty} \subset R_{+}^{1+m}$ of (2.1) and (3.1) resp.:

$$\sum_{t=1}^{T} \pi^{t}p[x(t);t] = u(1)'A[x(0);0] + \sum_{t=1}^{T} \{\mu(t)+f(t)'u(t)\} -$$

$$- \sum_{t=1}^{T} u(t)'y(t) - \sum_{t=1}^{T} v[u(t),u(t+1),x(t),\mu(t);t] -$$

$$- u(T+1)'A[x(T);T], \quad T = 1,2,...(3.6)$$

With the help of these equalities it can be demonstrated that the definitions of superiority, and the specifications 2-a, to 2-f, imply a number of properties concerning superiority.

4. A sufficient condition for superiority.

Now, we consider the case that the functions $A[\cdot;t]$ and $B[\cdot;t]$ satisfy at least one of the following conditions:

$$A[x+y;t] \geqq A[y;t], \text{ for all } x,y \in R_{+}^{n}, \; t = 1,2,..., \qquad (4.1)$$

$$B[x+y;t] \geqq B[y;t], \text{ for all } x,y \in R_{+}^{n}, \; t = 1,2,..., \qquad (4.2)$$

Under these conditions one can deduce that the equalities of (3.6) imply, for every pair of feasible solutions $\{(x(t),y(t))\}_1^\infty$ and $\{(\mu(t),u(t))\}_1^\infty$ of the <u>primal problem</u> (i.e. the problem of §2) and the dual problem, the following properties:

a) If the functions $A[\cdot;t]$ satisfy (4.1):

$$\sum_{t=1}^{T} \pi^t p[x(t);t] \leqq u(1)'A[x(0);0] + \sum_{t=1}^{T} \{\mu(t)+f(t)'u(t)\},\ T = 1,2,\ldots.$$

b) If the functions $B[\cdot;t]$ satisfy (4.2):

$$\sum_{t=1}^{T} \pi^t p[x(t);t] \leqq u(1)'A[x(0);0] + \sum_{t=1}^{T+1} \{\mu(t)+f(t)'u(t)\},\ T = 1,2,\ldots.$$

c) If at least one of the conditions (4.1) or (4.2) is satisfied, then feasible solutions $\{(\hat{x}(t),\hat{y}(t))\}_1^\infty$ and $\{(\hat{\mu}(t),\hat{u}(t))\}_1^\infty$ of the primal and the dual problem for which the sequences of objective functions are convergent, and for which:

$$\left.\begin{array}{l} \hat{u}(t)'\hat{y}(t) = 0,\ v[\hat{u}(t),\hat{u}(t+1),\hat{x}(t),\hat{\mu}(t);t] = 0,\ t=1,2,\ldots \\ \hat{u}(t+1)'A[\hat{x}(t);t] \to 0,\ \text{for } t \to \infty \end{array}\right\}, \tag{4.3}$$

are both superior solutions.

Motivation with respect to property (c): By virtue of (3.6) the conditions concerning the feasible solutions imply:

$$\lim_{T\to\infty} \sum_{t=1}^{T} \pi^t p[\hat{x}(t);t] = \hat{u}(1)'A[x(0);0] + \lim_{T\to\infty} \sum_{t=1}^{T} \{\hat{\mu}(t)+f(t)'\hat{u}(t)\}. \tag{4.4}$$

By property (a) or by property (b), the latter excludes the existence of feasible solutions $\{(\bar{x}(t),\bar{y}(t))\}$ and $\{(\bar{\mu}(t),\bar{u}(t))\}$ as mentioned in the definitions of superiority (2.4) and (3.4).

Next paragraphe shall give a very special feasible solution which satisfy the sufficient conditions for superiority of (c).

5. Balanced superior solutions for time-invariant programming problems.

The programming problem of §2 is called time-invariant if it can be written in the form:

$$\left.\begin{array}{l} B[x(1)]+y(1) = f+A[x(0)] \\ \left.\begin{array}{l} B[x(t+1)]-A[x(t)]+y(t+1) = f \\ x(t),y(t) \geqq 0 \end{array}\right\} t = 1,2,\ldots \end{array}\right\}, \tag{5.1}$$

with the sequence of objective functions:

$$\left\{\sum_{t=1}^{T} \pi^t p[x(t)]\right\}_{T=1}^{\infty} . \tag{5.2}$$

Now, the corresponding dual problem of §3 takes the form:

$$\left.\begin{array}{l} u(t)'B[z]-u(t+1)'A[z]+\mu(t) \geqq \pi^t p[z], \text{ for all } z \in R^n_+ \\ u(t) \geqq 0 \end{array}\right\} t = 1,2,\ldots, \tag{5.3}$$

with the objective functions:

$$u(1)'A[x(0)] + \sum_{t=1}^{T} \{\mu(t)+f'u(t)\}, \ T = 1,2,\ldots \tag{5.4}$$

The functions $A[\cdot]$, $B[\cdot]$, and $p[\cdot]$ are supposed to satisfy the conditions 2-c, d, f.

For these programming problems non-negative solutions $(\tilde{x},\tilde{y}) \in R^{n+m}_+$, $(\tilde{\mu},\tilde{u}) \in R^{1+m}_+$ of the system:

$$\left.\begin{array}{l} B[x]-A[x]+y = f \\ u'\{B[z]-\pi A[z]\}+\mu \geqq p[z], \text{ for all } z \in R^n_+ \\ u'y = 0, \ u'\{B[x]-\pi A[x]\}+\mu = p[x] \end{array}\right\}, \tag{5.5}$$

are very interesting. For putting the initial vector $x(0) := \tilde{x}$, it appears that

$$\left.\begin{aligned} (x(t),y(t)) &:= (\tilde{x},\tilde{y}) \\ (\mu(t),u(t)) &:= \pi^t(\tilde{\mu},\tilde{u}) \end{aligned}\right\} t = 1,2,..., \tag{5.6}$$

are feasible solution of (5.1) and (5.2), which satisfy the sufficient condition for superiority mentioned in §4.
The non-negative solution $(\tilde{x},\tilde{y})$, $(\tilde{\mu},\tilde{u})$ will be called an <u>equilibrium combination</u> and sequences generated by (5.6): <u>balanced superior (or optimal) solutions</u>.
The main result in this study can be formulated as follows:

If the programming problems consisting of (5.1), (5.2) and of (5.3), (5.4) satisfy the conditions:

a) $A[\cdot]$ and $p[\cdot]$ are concave; $B[\cdot]$ is convex;

b) $A[0] = 0$, $B[0] = 0$, $p[0] = 0$.

c) $A[\cdot]$, $B[\cdot]$, and $p[\cdot]$ are continuous.

d) For every x, $y \in R^n_+$: $A[x+y] \geqq A[x]$, or:
for every x, $y \in R^n_+$: $B[x+y] \geqq B[x]$.

e) The system $\{B[x]-\pi A[x] < f,\ x \geqq 0\}$ is solvable.

f) A number $\bar{\mu} \geqq 0$, vectors $\bar{u} \geqq 0$, $\bar{v} > 0$ exist, such that:
$\bar{u}'\{B[z]-A[z]\}+\bar{\mu} \geqq \underbrace{\bar{v}'z}_{p[z]}$, for all $z \in R^n_+$.

Then an equilibrium combination exists, The proof is constituted by §6 to §9. A numerical example can be found in §12.

<u>6. Reduction to a convex programming problem in a finite dimensional space.</u>

For any vector $w \in R^m_+$, we consider the following convex programming problem in a finite dimensional Eudiclean space:

$$\phi[w] := \sup_{(x,y)} q[x;w] \mid B[x]-A[x]+y = f,\ x,\ y \geqq 0, \tag{6.1}$$

and its dual problem:

$$\psi[w] := \inf_{(\mu,u)} \mu+f'u \left| \begin{array}{l} u'\{B[z]-A[z]\}+\mu \geqq q[z;w], \text{ for all } z \geqq 0 \\ \mu,\ u \geqq 0, \end{array} \right. \tag{6.2}$$

where the functions $A[\cdot]$, $B[\cdot]$ satisfy the conditions 5-a to 5-c, and where $q[\cdot;\cdot]: R_+^{n+m} \to R$ is supposed to be concave, continuous, and such that for all $w \in R_+^m$: $q[0;w] = 0$.
We call the problems (6.1), (6.2) <u>regular</u> for some $w \in R_+^m$ if: (6.1) possesses a feasible solution x,y with $y > 0$, and if, in addition, a number $\bar{\mu} \geqq 0$ and vectors $\bar{u} \geqq 0$, $\bar{v} > 0$ exist satisfying.

$$\bar{u}'\{B[z]-A[z]\}+\bar{\mu} - q[z;w] \geqq \bar{v}'z, \text{ for all } z \geqq 0 \tag{6.3}$$

Now, we can memorate some well known properties:

a) If the problems (6.1) and (6.2) regular for some $w \in R_+^m$, then: $\phi[w] = \psi[w]$, the problems both possess optimal solutions, moreover: feasible solution $(\hat{x},\hat{y})$, $(\hat{\mu},\hat{u})$ of (6.1) and (6.2) both are optimal if, and only if: $\hat{u}'\hat{y} = 0$, $\hat{u}'\{B[\hat{x}]-A[\hat{x}]\}+\hat{\mu} = q[\hat{x};w]$.

b) Let $W \subset R_+^m$ be a compact set such that $\text{int}(W) \neq \emptyset$ and such that, for all $w \in W$, the problems (6.1) and (6.2) are regular. Denoting the power set of a set S by $\Pi(S)$, let $\overline{MU}: W \to \Pi(R^{1+m})$ be the multi-function defined by:

$$\overline{MU}[w] := \left\{ (\mu,u) \in R_+^{1+m} \left| \begin{array}{l} u'\{B[z]-A[z]\}+\mu \geqq q[z;w],\ z \in R_+^n \\ \mu+f'u = \psi[w] \end{array} \right. \right\}. \tag{6.4}$$

Then this multi-function is upper semicontinuous on W. (viz. ref. 1, page 109-116). Moreover, for every $w \in W$ is $\overline{MU}[w]$ convex and non-empty. Note: $\overline{MU}[w]$ is the set of optimal solutions of (6.2) belonging to W.

From now on, we shall specify the function $q[\cdot;\cdot]$ by:

$$q[x;w] := p[x]-(1-\pi)w'A[x],\ (x,w) \in R_+^{n+m}, \tag{6.5}$$

$p[\cdot]$ being the original objective function satisfying 5-a, b, c.

7. Proposition.

System 5.5, which satisfies 5-a, b, c, possesses a non-negative solution if, and only if, a $\hat{w} \in R^m_+$ exists, such that the corresponding problems (6.1), (6.2) with $q[\cdot;\cdot]$ specified by (6.5), possesses an optimal solution $(\hat{x},\hat{y})$, $(\hat{\mu},\hat{u})$, satisfying $\hat{u} = w$.

Proof Necessary: Let $(\tilde{x},\tilde{y})$, $(\tilde{\mu},\tilde{u})$ be a non-negative solution of (5.5). Then, the definitions (5.5), (6.1), (6.2) and (6.5) and property 6-a imply: for $w := \tilde{u}$, the vectors $(\tilde{x},\tilde{y})$, $(\tilde{\mu},\tilde{u})$ are optimal solutions.
Sufficient: Let $(\hat{x},\hat{y})$, $(\hat{\mu},\hat{u})$ be optimal solutions of (6.1) and (6.2) with $q[\cdot;w]$ specified by $p[\cdot]-(1-\pi)\hat{u}'A[\cdot]$. Then, $(\hat{\mu},\hat{u})$ satisfies $\hat{u}\{B[z]-\pi A[z]\}+\hat{\mu} \geqq p[z]$, for all $z \in R^n_+$. Moreover, the optimality implies, by virtue of property 6-a: $\hat{u}'\hat{y} = 0$, $\hat{u}'\{B[\hat{x}]-\pi A[\hat{x}]\}+\hat{\mu}-p[\hat{x}] = \hat{u}'\{B[\hat{x}]-A[\hat{x}]\}+\hat{\mu}-p[\hat{x}]+(1-\pi)\hat{u}'A[\hat{x}] = 0$. Thus, it appears that $(\hat{x},\hat{y})$, $(\hat{\mu},\hat{u})$ is a non-negative solution of 5.5.

8. Proposition.

Suppose the programming problems (6.1), (6.2) with $q[\cdot;\cdot]$ specified by (6.5) satisfy 5-a, b, c. Then, the existence of a set $W \subset R^m_+$ satisfying:

a) W is compact, convex, and $\text{int}(W) \neq \emptyset$.

b) For every $w \in W$, the problems (6.1), (6.2) are regular.

c) For every $w \in W$: $\overline{MU}[w] \subset R^1 \times W$. (viz. def. 6.4),

implies the existence of a non-negative solution of 5.5.

Proof: By virtue of property 6-b, supposition (b) implies:

(1) $\overline{MU}[\cdot] : W \to \Pi(R^{1+m})$ is upper semicontinuous on W.

(2) For every $w \in W$: $\overline{MU}[w]$ is non-empty and convex.

Defining the multi-function $\overline{U}[\cdot] : W \to \Pi(R^m)$ by:

$$\overline{U}[w] := \{u \in R^m_+ | R^1 \times \{u\} \cap \overline{MU}[w] \neq \emptyset\}, \tag{8.1}$$

it is clear that (1) and (2) imply:

(3) $\overline{U}[\cdot] : W \to \Pi(R^m)$ is upper semicontinuous on W.

(4) For every $w \in W$: $\overline{U}[w]$ is non-empty and convex.

Moreover, supposition (c) and definition (8.1) imply:

(5) For every $w \in W$: $\overline{U}[w] \subset W$.

Thus, by virtue of Kakutani's fixed point theorem (ref. 1, page 174) the properties (3), (4) and supposition (a) imply the existence of a $\tilde{w} \in W$ such that $\tilde{w} \in \overline{U}[\tilde{w}]$, and so, by virtue of proposition 7 and the definitions (6.4) and (8.1), the existence of a feasible solution of (5.5), as well.

9. Theorem.

If system (5.5) (and so the programming problem 5.1 to 5.4) satisfies the conditions 5-a to 5-f, then an equilibrium combination exists, and so a balanced superior solution for programming problem (5.1) to (5.4) as well.

Proof: First we assume that 5-d is satisfied by the function $A[\cdot]$. With this assumption, we shall construct a set $W \subset R^m_+$ which satisfies the conditions formulated in proposition 8.

Let $\underline{x} \geq 0$, $\underline{y} > 0$, satisfy $B[\underline{x}] - \pi A[\underline{x}] + \underline{y} = f$ (supposition 5-e), then: for every combination (w, μ_w, u_w) such that $w \in R^m_+$, $(\mu_w, u_w) \in \overline{MU}[w]$ ($\overline{MU}[\cdot]$ defined by 6.4):

$$f'u_w = u_w'\{B[\underline{x}]-\pi A[\underline{x}]+\underline{y}\} =$$

$$= u_w'\{B[\underline{x}]-A[\underline{x}]\}+u_w'\{\underline{y}+(1-\pi)A[\underline{x}]\} \geqq$$

$$\geqq p[\underline{x}]-(1-\pi)w'A[\underline{x}]+u_w'\{\underline{y}+(1-\pi)A[\underline{x}]\}-\mu_w \tag{9.1}$$

Let $\underline{u} \geqq 0$, $\underline{v} > 0$ be vectors and let $\underline{\mu} \geqq$ be a number such that $\underline{u}'\{B[z]-A[z]\}-p[z]+\underline{\mu} \geqq \underline{v}'z$, for all $z \in R_+^n$ (supposition 5-f). Then, the supposition 5-d concerning $A[\cdot]$ implies that

$$\underline{u}'\{B[z]-A[z]\}+\underline{\mu}-p[z]+(1-\pi)w'A[z] \geqq \underline{v}'z, \text{ for all } w \in R_+^m \text{ and all } z \in R_+^n. \tag{9.2}$$

So $(\underline{\mu},\underline{u})$ is a feasible solution for all programming problems (6.2) with $w \in R_+^m$. This implies:

$$\mu_w+f'u_w \leqq \mu+f'\underline{u} \text{ for all } (\mu_w,u_w) \in \overline{MU}[w],\ w \in R_+^m$$

Combining the latter inequality with (9.1) we find:

$$\underline{\mu}+f'u \geqq p[\underline{x}]-(1-\pi)w'A[\underline{x}]+u_w'\{\underline{y}+(1-\pi)A[\underline{x}]\}, \tag{9.3}$$

for all $(\mu_w,u_w) \in \overline{MU}[w]$.

Let ε be a positive number (small enough) such that $\underline{y}+(1-\pi)A[\underline{x}] \geqq (1+\varepsilon)(1-\pi)A[\underline{x}]$ (this is possible by $\underline{y} > 0$). Defining $\bar{z}:= \underline{y}+(1-\pi)A[\underline{x}]$, inequality (9.3) implies:

$$\bar{z}'u_w \leqq \frac{z'w}{1+\varepsilon} + \underline{\mu}+f'\underline{u}-p[\underline{x}], \tag{9.4}$$

for all $u_w \in \underline{U}[w]$, $w \in R_+^m$.

Defining the set

$$W:= \{w \in R_+^m \mid \bar{z}'w \leqq \frac{1+\varepsilon}{\varepsilon}(\underline{\mu}+f'\underline{u}-p[\underline{x}])\}, \tag{9.5}$$

inequality (9.4) implies

$$\bar{z}'u_w \leqq (\frac{1}{\varepsilon} + 1)(f'\underline{u} - p[\underline{x}]), \text{ for all } (\mu_w, u_w) \in \overline{MU}[w], \qquad (9.6)$$

provided $w \in W$. Concerning the set W in (9.5) we have:

a) For every $w \in W$: $\overline{MU}[w] \subset R^1 \times W$.

Since $\underline{y} > 0$, $(1-\pi)A[\underline{x}] \geqq 0$, the definitions of $\bar{z}$ and W imply:

b) W is compact, convex, and $\text{int}(W) \neq \emptyset$.

The suppositions 5-e, f, and the suppositions 5-b, d concerning $A[\cdot]$, imply:

c) For every $w \in W$ the problem (6.1), (6.2), with $q[\cdot;\cdot]$ specified by (6.5) are regular.

Thus, by virtue of proposition 8, we may conclude: system (5.5) possesses a non-negative solution.

Note; in case assumption 5-d is satisfied by function $B[\cdot]$, one can start from the programming problem:

$$\max \frac{1}{\pi} p[x] - (\frac{1}{\pi} - 1)w'B[x] \left| \begin{array}{l} B[x] - A[x] + y = f \\ x, y \geqq 0 \end{array} \right.$$

and its dual form:

$$\min \mu + f'u \left| \begin{array}{l} u'\{B[z] - A[z]\} + \mu \geqq \frac{1}{\pi}p[z] - (\frac{1}{\pi} - 1)w'B[z], \text{ for all } z \geqq 0 \\ u \geqq 0, \mu \geqq 0, \end{array} \right.$$

instead of (6.1) and (6.2). One may verify that the proof can be constructed in similar manner as in §7 to §9.

10. Kuhn-Tucker representation.

In many cases it is possible to express the conditions concerning the dual part of an equilibrium combination in terms of the derivatives of $A[\cdot]$, $B[\cdot]$, and $p[\cdot]$.

More precisly:

If a point $(\bar{x},\bar{y}) \in R_+^{n+m}$ satisfies the following conditions:

a) $B[\bar{x}] - A[\bar{x}] + \bar{y} = f$.

b) System $B[x] - \pi A[x] < f + (1-\pi)A[\bar{x}]$, $x \geqq 0$, is solvable.

c) The functions $A[\cdot]$, $B[\cdot]$, and $p[\cdot]$ are differentiable in $\bar{x}$; the derivatives in $\bar{x}$ are denoted: ∇A, ∇B, ∇p.

Then, $(\bar{x},\bar{y})$ is the primal part of an equilbrium combination if, and only if, an $(\bar{u},\bar{v}) \in R_+^{n+m}$ exists satisfying:

$$\left.\begin{array}{l} \bar{u}'\{\nabla B - \pi \nabla A\} - \bar{v}' = \nabla p \\ \\ \bar{u}'\bar{y} = 0, \ \bar{v}'\bar{x} = 0 \end{array}\right\}$$

Using the well known Kuhn-Tucker condition for optimality, this property follows from the fact that $(\bar{x},\bar{y})$ is the primal part of an equilbrium combination if, and only if, $(\bar{x},\bar{y})$ is an optimal solution of the programming problem $\{\sup p[x] \mid B[x] - \pi A[x] \leqq f + (1-\pi)A[\bar{x}],\ x \geqq 0\}$.

We observe that a dual part of the equilibrium combination in the sense of (5.5) can be given by $(\bar{\mu},\bar{u})$, where $\bar{\mu} := p[\bar{x}] - \bar{u}'\{B[\bar{x}] - A[\bar{x}]\}$.

11. Numerical aspects.

From §6 to §9 it appears that the existence of balanced superior solutions is proved by constructing a upper semicontinuous function $\underline{U}[\cdot]$ on a convex and compact set W in a finite dimentional space, such that $\underline{U}[W] \subset W$, in such a manner that the conditions of Kakutani's theorem are satisfied.
Moreover, the set W is defined by $W := \{w \in R_+^m \mid z'w \leqq M\}$, z being some positive finite dimensional vector and M being some positive number. That implies that the numerical methods developed by Scarf and Eaves (ref.10 and 4) can be used for the calculation of balanced superior solutions.

In two cases it is possible to formulate problem (5.5) as a linear complementarity problem which can be treated by the Lemke-Howson algorithm (ref. 2 and 9).

In the first case, all functions A[·] , B[·] , and p[·] are linear. Representing these functions by m×n-matrices A, B and by a n-dimensional vector p, problem (5.5) takes the form:

$$\left.\begin{aligned} &(B-A)x+y = f \\ &(B'-\pi A')u-v = p \\ &u'y = 0,\ v'x = 0 \end{aligned}\right\}. \qquad (11.1)$$

(Motivation: $(\mu,u) \in R_+^{1+m}$ satisfies: $u'(B-\pi A)z+\mu \geqq p'z$ for all $z \in R_+^n$, if, and only if: $u'(B-\pi A) \geqq p'$).

The corresponding linear complementarity problem can be formulated as the determination of vectors $(u,x) \in R_+^{m+n}$, $(y,v) \in R_+^{m+n}$ which satisfy:

$$\left.\begin{aligned} &\begin{bmatrix} 0 & (B-A) \\ -(B'-\pi A') & 0 \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} f \\ -p \end{bmatrix} \\ &(u,x)'(y,v) = 0 \end{aligned}\right\}. \qquad (11.2)$$

If the systems $\{(B-\pi A)x < f,\ x \geqq 0\}$ and $\{(B'-A') > p,\ u \geqq 0\}$ are solvable, and if, in addition, one of the matrices A, B are non-negative, then it can be shown that Lemke-Howson's algorithm will give us at least one equilibrium combination (ref. 3 and 6).

In the second case, the functions A[·] , B[·] are linear and the function p[·] is concave quadratic.
The latter implies that p[·] can be written in the form $q'x - \frac{1}{2}x'Qx$, q being a n-vector and Q a symmetric semi-positive definite n×n-matrix. Using the Kuhn-Tucker representation of

§10, and representing the functions $A[\cdot]$, $B[\cdot]$ by $m\times n$-matrices A, B, problem (5.5) takes the form:

$$\left.\begin{aligned} &(B-A)x+y = f \\ &u'(B-\pi A)-v' = q'-x'Q \\ &u'y = 0,\ v'x = 0 \end{aligned}\right\}. \tag{11.3}$$

Clearly, the corresponding linear complementarity problem can be written:

$$\left.\begin{aligned} &\begin{bmatrix} 0 & (B-A) \\ -(B'-\pi A') & -Q \end{bmatrix}\begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} f \\ -q \end{bmatrix} \\ &(u,x)'(y,v) = 0 \end{aligned}\right\}. \tag{11.4}$$

If the systems $\{(B-\pi A)x < f,\ x \geqq 0\}$ and $\{u'(B-A) > q,\ u \geqq 0\}$ are solvable and if, in addition, one of the matrices A, B is non-negative, then, with the help of the Lemke-Howson algorithm, at least one equilibrium can be found. This statement can be proved in a similar manner as in the linear case (ref. 3 and 6).

12. Example: one sector model with Cobb-Douglas production function.

We consider the very simple growth model of §1, consisting of the restrictions (1.7) and of the objective function (1.10). The quantities appearing in this model are specified as follows:

a) growth factor of population: $\rho := 1$.

b) production function: $F[k,l] := k^{\alpha}.l^{\beta}$ for all $k,l \geqq 0$, α, β being positive numbers such that: $\alpha+\beta = 1$.

c) The objective function $p[c]$ is supposed to be an increasing differentiable function of the consumption. For every amount of consumption $c \geqq 0$, the derivative shall be denoted by $\delta p[c]$. So these assumptions imply that for every $c > 0$: $\delta p[c] > 0$.

Identifying the components of a vector $x \in R^4_+$ by: $(x_1, x_2, x_3, x_4) := (c, z, k, l)$ (being: consumption, investment, and aggregate capital per worker and the fraction of productive workers. viz. §1) the primal conditions of an equilibrium combination: $\{B[x]-A[x]+y = f,\ x \geqq 0,\ y \geqq 0\}$, can be written:

$$\left.\begin{aligned} & x_1 + x_2 - x_3^\alpha x_4^\beta + y_1 = 0 \\ & -x_2 + (1-\mu)x_3 + y_2 = 0 \\ & x_4 + y_3 = 1 \\ & x_1, x_2, x_3, x_4, y_1, y_2, y_3 \geqq 0 \end{aligned}\right\}. \tag{12.1}$$

Defining $x_5 := x_3^\alpha \cdot x_4^\beta$, for every $x \in R^4_+$, $x > 0$ the derivatives of $B[\cdot]$, $A[\cdot]$ can be written, in matrix-form:

$$\nabla B\Big|_x = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{12.2}$$

$$\nabla A\Big|_x = \begin{bmatrix} 0 & 0 & \alpha x_5/x_3 & \beta x_5/x_4 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{12.3}$$

Using the Kuhn-Tucker representation, the dual condition: $(\nabla B' - \pi \nabla A')u - v = \nabla p'$, can be written:

$$\left.\begin{aligned}
u_1 - v_1 &= \delta p[x_1] \\
u_1 - u_2 - v_2 &= 0 \\
-\pi\alpha(x_5/x_3)u_1 + (1-\pi\mu)u_2 - v_3 &= 0 \\
-\pi\beta(x_5/x_4)u_1 + u_3 - v_4 &= 0 \\
u_1,\ldots,u_3,v_1,\ldots,v_4 &\geqq 0
\end{aligned}\right\} . \tag{12.4}$$

The complementarity relations $u'y = 0$, $v'x = 0$ become:

$$u_i y_i = 0, \quad i = 1,2,3. \tag{12.5}$$

$$v_j x_j = 0, \quad j = 1,..,4. \tag{12.6}$$

Looking for an equilibrium combination with a positive consumption $x_1 > 0$, the conditions (12.1) imply $x_3 > 0$, $x_4 > 0$, and $x_2 > 0$, and together with (12.6): $v_1 = 0$, $v_2 = 0$, $v_3 = 0$, $v_4 = 0$, as well.
Then, the first and the second equality of (12.4) imply: $u_2 = u_1 = \delta p[x_1]$. Since $\delta p[x_1] > 0$ for all $x_1 > 0$, the thirth equality of (12.4) and $u_2 = u_1 = \delta p[x_1]$, $v_3 = 0$ imply:

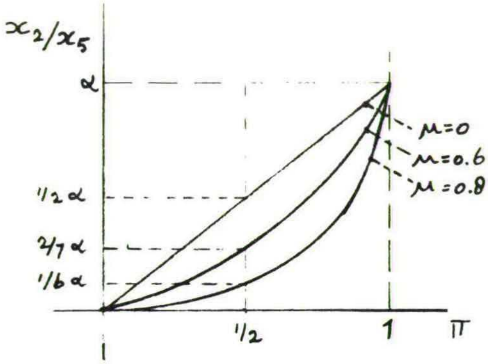
$$x_3 = \frac{\pi\alpha}{1-\pi\mu}\, x_5 . \tag{12.7}$$

Further, $u_1 > 0$, the fourth equality of (12.4), the relation $u_3 y_3 = 0$ (viz. 12.5), and the third equality of (12.1) imply successively: $u_3 > 0$, $y_3 = 0$, and $x_4 = 1$. Then, definition $x_5 := x_3^{\alpha} x_4^{\beta}$, and (12.7) imply:

$$x_3 = \left(\frac{\pi\alpha}{1-\pi\mu}\right)^{\frac{1}{\beta}} . \tag{12.8}$$

One may verify that, with the help of the latter relation all other quantities x_1, x_2, u_1, u_2, u_3 can be determined.

The quotient x_2/x_5 may be interpret as the faction of outputs that will be used for investents. From (12.7) and the second equality (with $y_2 = 0$) of (12.1) one can deduce:



$$\frac{x_2}{x_5} = \left(\frac{1-\mu}{1-\pi\mu}\right)\pi\alpha,$$

where $\mu \in [0,1]$ the coefficient of depreciation and $\pi \in]0,1[$ the time-discount factor.

We observe that similar results are found by Hansen and Koopmans (ref. 8) in case the functions A[·], B[·], and p[·] satisfy some special conditions, like linearity of A[·], B[·] and strict concavity of p[·].
Earlier, the existence of equilibrium combinations is demonstrated in case the functions A[·], B[·], p[·] are linear and the conditions 5-d, e, f are satisfied (ref. 5).

G.W.

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